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# Rough set models of some abstract algebras close to pre-rough algebra

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ABSTRACT

Rough set theory has already been algebraically investigated for decades and quasi-Boolean algebra has formed a basis for a number of structures emerging out of rough sets. Pre-rough algebra is one such algebra amongst them. A number of structures based on quasi-Boolean algebra but weaker than pre-rough algebra already exist. In this paper some algebras and their logics are added. Rough set models of the newly created algebras and some of the existing algebras are presented.

## 1. Introduction

Rough set theory was proposed by Pawlak [6] in 1982 to address information about objects in terms of some attribute–value systems. Thereafter, a huge amount of research has been carried out on the foundation and applications of the theory. Defining rough sets in various ways, several abstract algebraic structures have been developed. Pre-rough algebra, among them, is one (see Definition 2.1) [1].

The base of this abstract pre-rough algebra is a quasi-Boolean algebra (qBa) which is a structure $\langle U, \wedge, \vee, \neg, 0, 1\rangle$, where

1. $\langle U, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice,
2. $\neg\neg x = x$, for all $x$ in $U$,
3. $\neg(x \vee y) = \neg x \wedge \neg y$, for all $x, y$ in $U$.

A qBa is a more general structure than a Boolean algebra as the law of excluded middle ($x \vee \neg x = 1$) and the law of contradiction ($x \wedge \neg x = 0$) generally do not hold in a qBa. In [7], quasi-Boolean algebra has also been called De Morgan lattice.

Later, from different motivations, many abstract algebras stronger than qba but weaker than pre-rough algebra were developed [1,8,9,4,3]. Some of them are topological quasi-Boolean algebra (tqBa) [1], topological quasi-Boolean algebra with modal axiom $S_5$ (tqBa5)[8,12], intermediate algebra of type 1 (IA1), intermediate algebra of type 2 (IA2), intermediate algebra of type 3 (IA3) [12,9], System0 algebra, systemI algebra, systemII algebra [8].

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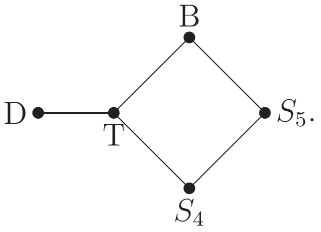
Pre-rough algebra has a rough set model in [1]. But, as per our knowledge, there is no "proper set theoretic rough set model" of the algebras tqBa, tqBa5, IA1, IA2, IA3, System0 algebra, SystemI algebra, SystemII algebra which are basically weaker than pre-rough algebras. The phrase 'proper set theoretic rough set model' means that it should be a set model and should not reduce to a pre-rough algebra. An attempt is taken in this paper to construct such models of the above algebras. As all these algebras are based on qBa, we focus our attention on a representation theorem of qBa presented by Rasiowa in [7]. Following her, for any set U we can construct a structure $\left\langle 2^U, \cap, \cup, \sim, \varnothing, U \right\rangle, 2^U$ representing the power set of $U$, which can be proved to be a quasi Boolean algebra, where $\sim$, called quasi-complementation, is not the standard set-theoretic complementation $^c$ but is defined by means of an involution $g$ (i.e. a map satisfying $g(g(u)) = u$, for all $u \in U$) namely $\sim P = (g(P))^c, P \subseteq U$. This qBa may be used as base to construct models of the said algebras where the quasi-complementation $\sim$ would be the counterpart of the quasi-negation $\neg$ available in the algebras. The next step is to construct the counterparts of the operators $I$ and $C$ present in the abstract algebras. These operators are dual with respect to $\neg$, i.e., $I = \neg C \neg$ and vice versa. We look for the counterparts of these operators $I$ and $C$. The standard lower and upper approximation operators in a generalised approximation space $\langle U, R \rangle$ are dual with respect to the set complementation [14,15,10] but not with respect to the quasi-complementation [11]. For this reason, we do not use these approximation operators as counterparts in the quasi-Boolean algebra $\left\langle 2^U, \cap, \cup, \sim, \varnothing, U \right\rangle$. So, the question is, how to define them in this quasi-Boolean algebra. In this regard, we follow our recent paper [11] where a new approximation space $\langle U, R^g \rangle$, called $g$-approximation space, has been defined out of a generalized approximation space $\langle U, R \rangle$, and an arbitrary involution $g$ on $U$. In this space, lower–upper approximations $\underline{P}_g$ and $\overline{P}^g$ of a set $P \subseteq U$ have been defined in order to make them dual with respect to the quasi-complementation. Concerning these approximations, rough set models of some of the aforesaid algebras may be constructed but the question of 'proper' that has been raised early remains unanswered. To solve the issue, we have studied the properties of algebras and $\underline{P}_g, \overline{P}^g$ in a $g$-approximation space $\langle U, R^g \rangle$. In case of the algebras, it is found that some standard modal axioms e.g., T:$Ix \leqslant x, S_4 : Ix \leqslant IIx$ [5] and hence D:$Ix \leqslant Cx$ are available in tqBa, whereas the modal axioms T, $S_4, S_5 : CIx \leqslant Ix$ and hence axioms D, B:$CIx \leqslant x$ are present in tqBa5, IA1, IA2 and IA3. But, no information is available regarding the algebraic counterpart of the modal axiom K in these algebras. We notice that the algebraic counterpart of K in the form $I(x \to y) \to (Ix \to Iy) = 1$ is irrelevant for some of the algebras tqBa, tqBa5, IA2, IA3 as there is no $\to$ obeying the property ($P_{\to}$) [for details see Section 2].

In [10], it has been shown that standard lower and upper approximations $\underline{P}_R$ and $\overline{P}^R$ in a generalized approximation space $\langle U, R \rangle$ satisfy the modal axiom K in the form $\underline{P^c \cup Q}_R \subseteq (\underline{P}_R)^c \cup \underline{Q}_R$. But, in our case, the lower–upper approximations $\underline{P}_g$ and $\overline{P}^g$ generally do not fulfill $\underline{\sim P \cup Q}_g \subseteq \sim (\underline{P}_g) \cup \underline{Q}_g$ whose algebraic form (in quasi-Boolean base) is $I(\neg x \vee y) \leqslant \neg Ix \vee Iy$. Hence, for a proper set theoretic rough set model, it is required to check whether this form of K: $I(\neg x \vee y) \leqslant \neg Ix \vee Iy$ holds in the above algebras or not. In SubSection 2.1, we have examined it and found that this form of K holds in pre-rough algebra, IA1, IA2 but does not hold in tqBa, tqBa5, IA3, System0, SystemI and SystemII.

We have defined a few abstract algebras viz. stqBa, stqBa-D, stqBa-T and stqBa-B based on qBa in Subsection 2.2. In each of these algebras, one or more than one standard modal axioms are added except this form of K (as K generally does not hold in our models) in order to develop algebraic systems in quasi-Boolean base parallel to the existing standard modal systems depicted below.



Further, these algebras are correlated with the old algebras and logics and proper set theoretic rough set models have been constructed for them.

For a clear understanding about the algebras we refer to Fig. 1 on page 5.

Section-wise details of this paper are as follows.

In Section 2, a number of algebraic structures based on qBa are defined in order to enhance the properties of $I$ in hierarchical order starting from modal axiom $D$. A diagram of a relationship between the earlier algebras and newly created algebras has been presented in Fig. 1. Availability of the modal axiom K:$I(\neg x \vee y) \leqslant \neg Ix \vee Iy$ has been checked for all the algebras mentioned in this section. Section 3 contains only sequent calculi corresponding to the new algebraic structures as no Hilbert-type axiomatic systems can be constructed for them. Section 4 deals with set models. In this section, proper set the-

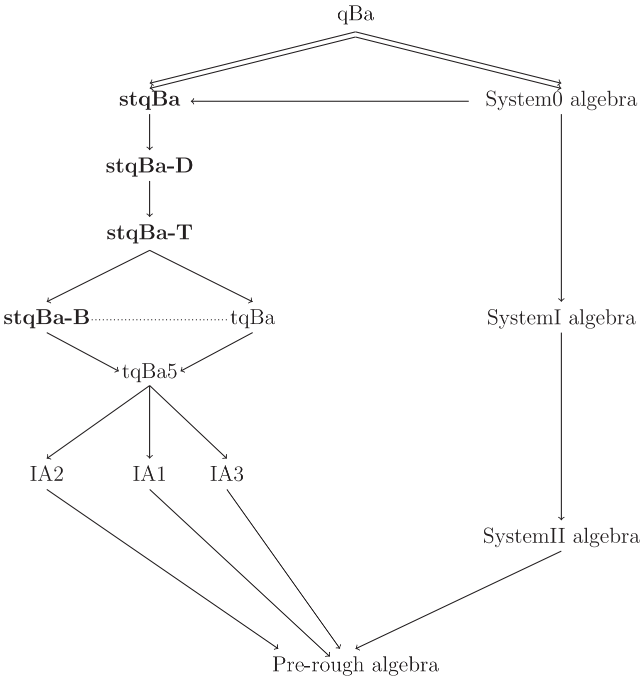


**Fig. 1.** Relationship diagram of the algebras mentioned in Section 2. Bold-faced algebras are newly introduced in this paper whereas others are available in different literature. $P \rightrightarrows Q$ stands for the algebra $Q$ contains one new operation and some axioms for the new operation than the algebra $P$. $P \longrightarrow Q$ stands for both the algebras $P$ and $Q$ have the same operations but $Q$ contains some more axioms than $P$. $P \cdots Q$ stands for the algebras $P$ and $Q$ are independent to each other.

oretic rough set models of some of the algebras mentioned in Section 2 have been developed. Some concluding remarks are included in the last Section 5.

## 2. Some algebraic structures based on qBa along with modal operators

In this section, a number of algebras stqBa, stqBa-D, stqBa-T and stqBa-B will be introduced. The reason for introducing these algebras is as follows:

These structures except stqBa make algebraic systems based on quasi-Boolean algebra (not Boolean algebra) corresponding to the standard modal systems D,T and B. The algebra stqBa also corresponds to a modal system but which is a non-standard one. The other standard modal systems $S_4$ and $S_5$ already correspond to the existing algebras tqBa and tqBa5. On the other hand, by introducing these algebras the gap with respect to the standard modal systems in the previous studies of algebras is filled up now.

Fig. 1 will make the motivation transparent.

As mentioned in the introduction, the availability of the modal axiom K in the form $I(\neg x \vee y) \leqslant \neg Ix \vee Iy$ will be checked for the old algebras. The definitions of these algebras are:

**Definition 2.1** [1]. A pre-rough algebra is a structure $\langle U, \wedge, \vee, \neg, I, 0, 1 \rangle$, where $I$ is a unary operator on $U$ with the following conditions:

1. $\langle U, \wedge, \vee, \neg, 0, 1 \rangle$ is a qBa.
2. $I1 = 1$.
3. $I(x \wedge y) = Ix \wedge Iy$, for all $x, y \in U$.
4. $Ix \leqslant x$, for all $x \in U$ ($\leqslant$is the lattice order).
5. $IIx = Ix$, for all $x \in U$.
6. $CIx = Ix$, for all $x \in U$, where $Cx = \neg I \neg x$.
7. $\neg Ix \vee Ix = 1$, for all $x \in U$.
8. $I(x \vee y) = Ix \vee Iy$, for all $x, y \in U$.
9. $Cx \leqslant Cy \mathrm{and} Ix \leqslant Iy \mathrm{imply} x \leqslant y$, for all $x, y \in U$.

**Definition 2.2.** An algebraic structure $\langle U, \wedge, \vee, \neg, I, 0, 1\rangle$ is called

(a) a topological quasi-Boolean algebra(tqBa)[9] if and only if it satisfies the conditions from 1 to 5 as stated in Definition 2.1 of pre-rough algebra,
(b) a topological quasi-Boolean algebra 5(tqBa5)[9] if and only if it is a tqBa + condition 6,
(c) an intermediate algebra of type 1 (IA1) [9] if and only if it is a tqBa5 + condition 7,
(d) an intermediate algebra of type 2 (IA2) [9] if and only if it is a tqBa5 + condition 8,
(e) an intermediate algebra of type 3 (IA3) [9] if and only if it is a tqBa5 + condition 9,
(f) a System0 algebra [8] if and only if it satisfies the conditions $1, 2$ and $x \leqslant y$implies$Ix \leqslant Iy$, for all $x, y \in U$,
(g) a SystemI algebra [8] if and only if it is a System0 algebra + condition 7 + condition 9,
(h) a SystemII algebra [8] if and only if it satisfies the conditions from 1 to 3 and 7, 9.

In pre-rough algebra, the rough implication $\rightarrow$: $x \rightarrow y \equiv (\neg Ix \vee Iy) \wedge (\neg Cx \vee Cy)$, for all $x, y$ was defined in terms of the other operations satisfying the property ($P_{\rightarrow}$):

$$x \leqslant y \text{ if and only if } x \rightarrow y = 1, \text{for all } x, y.$$

This $\rightarrow$ corresponds to the rough inclusion [1]. It is also crucial to the logical connective viz. implication ($\Rightarrow$) for developing pre-rough and rough logics. It is to be noted that an implication satisfying the property ($P_{\rightarrow}$) is required in an abstract algebra to develop the Hilbert-type axiomatic system corresponding to the algebra. This rough implication is also available in SystemI algebra and SystemII algebra [8].

However, it was established in [2,9] that such an operation $\rightarrow$can not be defined in terms of other operations fulfilling the property ($P_{\rightarrow}$) in tqBa, tqBa5, IA2, IA3 (see Example 2.1) generally. As a result, the Hilbert-type axiomatic system corresponding to these algebras can not be constructed. However, the Sequent Calculi have been developed corresponding to these algebras in [12,8].

**Example 2.1.** (See[2,9]) A four-element tqBa as well as tqBa5, IA2 and IA3 is considered whose lattice structure follows Fig. 2. The unary operation $\neg$ is defined as $\neg x = x, \neg y = y, \neg 1 = 0, \neg 0 = 1$. $I$ is defined as the identity operator, i.e., $Iz = z$, for all $z$. Now $x \rightarrow x$ will be an element involving $x, \neg, \wedge, \vee$ and $I$. But, this example shows that $\neg x = x, x \wedge x = x, x \vee x = x, Ix = x$ and hence $x \rightarrow x = x(\neq 1)$ while $x \leqslant x$.

In IA1, the availability of such $\rightarrow$is still open.

### *2.1. Availability of modal axiom K in the aforesaid algebras*

We now examine whether the algebraic counterpart of the modal axiom K in the form $I(\neg x \vee y) \leqslant \neg Ix \vee Iy$ is available or not in the above algebras.

**Proposition 2.1.** $I(\neg x \vee y) \leqslant \neg Ix \vee Iy$ holds in a pre-rough algebra.

**Proof.**

$$\begin{aligned}&I(\neg x \vee y) = I\neg x \vee Iy\\&\leqslant \neg x \vee Iy\\&\leqslant \neg Ix \vee Iy (\text{as } Ix \leqslant x)\end{aligned}$$

**Proposition 2.2.** $I(\neg x \vee y) \leqslant \neg Ix \vee Iy$ holds in any IA2.

**Proposition 2.3.** $I(\neg x \vee y) \leqslant \neg Ix \vee Iy$ holds in any IA1.

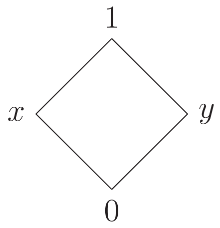


**Fig. 2.** Hasse diagram (tqBa, tqBa5, IA2, IA3).

**Proof.** First we shall show that in any IA1, $I(x \vee Iy) = Ix \vee Iy$ holds, for all $x, y$.

$$\begin{aligned}
I(x \vee Iy) &= I(x \vee Iy) \vee \{\neg Iy \wedge I(x \vee Iy)\} \\
&= I(x \vee Iy) \vee \{I\neg Iy \wedge I(x \vee Iy)\} (\text{as } CIz = Iz \text{ gives } I\neg Iz = \neg Iz) && (1) \\
&= I(x \vee Iy) \vee I\{\neg Iy \wedge (x \vee Iy)\} \\
&= I(x \vee Iy) \vee \neg I\neg I\{\neg Iy \wedge (x \vee Iy)\} (\text{as } CIz = Iz) \\
&= I(x \vee Iy) \vee \neg I\neg\{\neg Iy \wedge I(x \vee Iy)\} \\
&= I(x \vee Iy) \vee \neg I\{Iy \vee \neg I(x \vee Iy)\} \\
&= Iy \vee I(x \vee Iy) \vee \neg I\{Iy \vee \neg I(x \vee Iy)\} (\text{as } Iy \leqslant I(x \vee Iy)) \\
&= \{Iy \vee \neg I\{Iy \vee \neg I(x \vee Iy)\} \vee I(x \vee Iy)\} \wedge 1 \\
&= \{Iy \vee \neg I\{Iy \vee \neg I(x \vee Iy)\} \vee I(x \vee Iy)\} \\
&\wedge \{(Iy \vee \neg I(x \vee Iy)) \vee \neg I(Iy \vee \neg I(x \vee Iy))\} \\
&= \{Iy \vee \neg I(Iy \vee \neg I(x \vee Iy))\} \vee \{I(x \vee Iy) \wedge \neg I(x \vee Iy)\} \\
&= \{Iy \vee \neg I(Iy \vee \neg I(x \vee Iy))\} \\
&= \{Iy \vee \neg I\neg(\neg Ix \wedge I(x \vee Iy))\} \\
&= \{Iy \vee \neg I\neg I(\neg Iy \wedge (x \vee Iy))\} by 2.1 && (2) \\
&= \{Iy \vee \neg I\neg I\{(\neg Iy \wedge x) \vee (\neg Iy \wedge Iy)\}\} \\
&= \{Iy \vee \neg I\neg I(\neg Iy \wedge x)\} \\
&= \{Iy \vee I(\neg Iy \wedge x)\} (\text{as } CIz = Iz) && (3) \\
&= \{Iy \vee (\neg Iy \wedge Ix)\} by 2.1 && (4) \\
&= (Iy \vee \neg Iy) \wedge (Iy \vee Ix) \\
&= Ix \vee Iy && (5)
\end{aligned}$$

Proof of the main result:

$$\begin{aligned}
&I(\neg x \vee y) \vee (\neg Ix \vee Iy) \\
&= \{I(\neg x \vee y) \vee \neg Ix\} \vee Iy \\
&= I\{(\neg x \vee y) \vee \neg Ix\} \vee Iy \quad (by\ 2.1 and\ 2.2) \\
&= I\{(\neg x \vee \neg Ix) \vee y\} \vee Iy \\
&= I(\neg Ix \vee y) \vee Iy \\
&= \neg Ix \vee Iy \vee Iy \ (by\ 2.1 and\ 2.2) \\
&= \neg Ix \vee Iy
\end{aligned}$$

The following example is considered to show that $I(\neg x \vee y) \leqslant \neg Ix \vee Iy$ does not hold in a IA3.

**Example 2.2.** Let $P = \{0, x, y, 1\}$. Hasse diagram of the lattice follows Fig. 2. Operations $\neg$ and $I$ are defined in the tables given below.

| $\neg$ | $0$ | $x$ | $y$ | $1$ |
|---|---|---|---|---|
| | $1$ | $x$ | $y$ | $0$ |

| | $0$ | $x$ | $y$ | $1$ |
|---|---|---|---|---|
| $I$ | $0$ | $x$ | $0$ | $1$ |
| $C$ | $0$ | $x$ | $1$ | $1$ |

Clearly, $P$ with the above operations is a IA3. As $I(\neg x \vee y) = 1 \nleqslant \neg Ix \vee Iy = x$, the algebraic counterpart of K is not available in this IA3. Note that IP2 is also not valid in this algebra as $I(x \vee y) = 1 \neq Ix \vee Iy = x$.

**Remark 2.1.** By the above Example 2.2, it is clear that $I(\neg x \vee y) \leqslant \neg Ix \vee Iy$ does not hold in a tqBa and tqBa5.

The following example shows that $I(\neg x \vee y) \leqslant \neg Ix \vee Iy$ is not valid in a SystemII algebra.

**Example 2.3.** Let $Q = \{0, x, 1\}$. Hasse diagram of the lattice is given in Fig. 3 and operations $\neg$ and $I$ are defined in the tables given below.

| $\neg$ | $0$ | $x$ | $1$ |
|---|---|---|---|
| | $1$ | $x$ | $0$ |

| | $0$ | $x$ | $1$ |
|---|---|---|---|
| $I$ | $0$ | $1$ | $1$ |
| $C$ | $0$ | $0$ | $1$ |

$Q$ with the above operations is a SystemII algebra. The algebraic counterpart of K is not valid in this algebra as $I(\neg x \vee 0) = 1 \nleqslant \neg Ix \vee I0 = 0$.

**Remark 2.2.** As a SystemII algebra is a SystemI algebra as well as a System0 algebra, by Example 2.3, it is clear that K: $I(\neg x \vee y) \leqslant \neg Ix \vee Iy$ does not hold in a SystemI algebra and System0 algebra.

**Fig. 3.** Hasse diagram(SystemII algebra).

### *2.2. Some new structures based on quasi-Boolean algebra along with standard modal axioms*

We now introduce the algebras stqBa, stqBa-D, stqBa-T and stqBa-B as follows.

**Definition 2.3.** An abstract algebra $\langle U, \wedge, \vee, \neg, I, 0, 1\rangle$ will be called a semi topological quasi-Boolean algebra (stqBa) if and only if

1. $\langle U, \wedge, \vee, \neg, 0, 1\rangle$ is a qBa.
2. $I1 = 1$.
3. $I(x \wedge y) = Ix \wedge Iy$, for all $x, y \in U$.

Example 2.3 is an instant of a stqBa where K: $I(\neg x \vee y) \leqslant \neg Ix \vee Iy$does not hold.
The following Example 2.4 shows a stqBa where the modal axiom D ($Ix \leqslant Cx, C \equiv \neg I \neg$) and T ($Ix \leqslant x$) do not hold.

**Example 2.4.** Let $S = \{0, 1\}$. Hasse diagram of the lattice is given in Fig. 4 and operations $\neg$ and $I$ are defined in the tables given below.

| $\neg$ | 0 | 1 |
|---|---|---|
| | 1 | 0 |

| | 0 | 1 |
|---|---|---|
| $I$ | 1 | 1 |
| $C$ | 0 | 0 |

$S$ with the above operations is a stqBa. The modal axioms D and T are not available in this algebra as $I0 = 1 \nleqslant C0 = 0$ and $I0 = 1 \nleqslant 0$.

**Definition 2.4.** Let $\langle U, \wedge, \vee, \neg, I, 0, 1\rangle$ be a stqBa. Then it is said to be a semi topological quasi-Boolean algebra with modal axiom D (stqBa-D) if and only if

1. $Ix \leqslant Cx$, for all $x \in U$ ($Cx = \neg I \neg x$).

The following example is a stqBa-D where the modal axiom T is not available.

**Example 2.5.** For this example, the same set $P$ as well as the same Hasse diagram of the lattice in Example 2.2 is considered. Operations $\neg$ and $I$ are defined in the tables given below.

| $\neg$ | 0 | $x$ | $y$ | 1 |
|---|---|---|---|---|
| | 1 | $x$ | $y$ | 0 |

| | 0 | $x$ | $y$ | 1 |
|---|---|---|---|---|
| $I$ | 0 | $y$ | $x$ | 1 |
| $C$ | 0 | $y$ | $x$ | 1 |

**Fig. 4.** Hasse diagram (stqBa).

Clearly, $P$ with the above operations is a stqBa-D. T does not hold in this algebra as $Ix = y \nleqslant x$.

**Definition 2.5.** Let $\langle U, \wedge, \vee, \neg, I, 0, 1\rangle$ be a stqBa. Then it is said to be a semi topological quasi-Boolean algebra with modal axiom T (stqBa-T) if and only if $Ix \leqslant x$, for all $x \in U$.

**Proposition 2.4.** A stqBa-T is always a stqBa-D.

**Remark 2.3.** By Example 2.5, it is clear that the converse of Proposition 2.4 is not true, i.e., a stqBa-D is not necessarily a stqBa-T.

The following example is a stqBa-T where the modal axioms B($CIx \leqslant x$) and $S_4$($Ix \leqslant IIx$) do not hold.

**Example 2.6.** Let $T = \{0, x, y, z, 1\}$. Hasse diagram of the lattice is given in Fig. 5 and operations ¬ and $I$ are defined in the tables given below.

| $\neg$ | $0$ | $x$ | $y$ | $z$ | $1$ |
|---|---|---|---|---|---|
| | $1$ | $z$ | $y$ | $x$ | $0$ |

| | $0$ | $x$ | $y$ | $z$ | $1$ |
|---|---|---|---|---|---|
| $I$ | $0$ | $0$ | $x$ | $x$ | $1$ |
| $C$ | $0$ | $z$ | $z$ | $1$ | $1$ |

$T$ with the above operations is a stqBa-T. The modal axioms B and $S_4$ are not valid in this algebra as $CIy = z \nleqslant y$ and $Iy = x \nleqslant IIy = 0$.

**Definition 2.6.** Let $\langle U, \wedge, \vee, \neg, I, 0, 1\rangle$ be a stqBa-T. Then it is said to be a semi topological quasi-Boolean algebra with modal axiom B (stqBa-B) if and only if $CIx \leqslant x$, for all $x \in U$ ($Cx = \neg I \neg x$).

The following example is a stqBa-B where the modal axioms $S_4$ and $S_5$($CIx \leqslant Ix$) do not hold.

**Example 2.7.** The same set $P$ in Example 2.2 is considered for this case also. Hasse diagram of the lattice is given in Fig. 6 and operations ¬ and $I$ are defined in the tables given below.

| $\neg$ | $0$ | $x$ | $y$ | $1$ |
|---|---|---|---|---|
| | $1$ | $y$ | $x$ | $0$ |

| | $0$ | $x$ | $y$ | $1$ |
|---|---|---|---|---|
| $I$ | $0$ | $0$ | $x$ | $1$ |
| $C$ | $0$ | $y$ | $1$ | $1$ |

$P$ with the above operations is a stqBa-B. $S_4$ and $S_5$ are not available in this algebra as $Iy = x \nleqslant IIy = 0$ and $CIy = y \nleqslant Iy = x$.

The following example is an instance of a tqBa where B and $S_5$ are not valid. Note that the modal axiom $S_4$ holds in a tqBa.

**Example 2.8.** The same set $P$ as well as the same Hasse diagram of the lattice in Example 2.2 is considered for this case also. Operations ¬ and $I$ are defined in the tables given below.

| $\neg$ | $0$ | $x$ | $y$ | $1$ |
|---|---|---|---|---|
| | $1$ | $y$ | $x$ | $0$ |

| | $0$ | $x$ | $y$ | $1$ |
|---|---|---|---|---|
| $I$ | $0$ | $x$ | $0$ | $1$ |
| $C$ | $0$ | $1$ | $y$ | $1$ |

$P$ with the above operations is a tqBa. But, B and $S_5$ are not available in this algebra as $CIx = 1 \nleqslant x = Ix$.

**Remark 2.4.** By Example 2.7 and Example 2.8, it is obvious that stqBa-B and tqBa are independent algebras.

Logics and proper rough set models of these newly created algebras have been presented in Section 3 and Section 4 respectively.

## 3. Sequent calculi

It has been mentioned in Section 2 that to develop the Hilbert-type axiomatic systems corresponding to the new algebras an implication (→) satisfying the condition ($P_{\rightarrow}$) has to have in the algebras.

We now check whether such a → is available or not in the newly created algebras stqBa, stqBa-D, stqBa-T and stqBa-B.

It is clear that Example 2.1 becomes an instance for the algebras stqBa, stqBa-D, stqBa-T and stqBa-B. As a result, no such → can be defined in terms of other operations satisfying the property ($\mathrm{P}_{\rightarrow}$) in these algebras. Hence, the Hilbert-type axiomatic systems corresponding to these algebras can not be constructed. However, the sequent calculi for these algebras will be presented below.

Sequent calculus is a system that is used to formulate a logic. It differs from Hilbert-type axiomatic systems and Natural Deduction. Sequent calculus deals with sequents which are of the form $A \Rightarrow B$, where $A$ and $B$ are finite multisets (possibly empty) of well formed formulas (wffs). We write, as conventionally, $\gamma, \beta, \gamma \Rightarrow \delta, \gamma, \delta, \beta$ in lieu of the sequent $\{\gamma, \beta, \gamma\} \Rightarrow \{\delta, \gamma, \delta, \beta\}$, where $\beta, \gamma, \delta$are wffs. It is to be noted that $\Rightarrow$ is just a symbol representing a sequent but is not available in the logic language.

The sequent calculi for the algebras stqBa, stqBa-D, stqBa-T and stqBa-B are named SCstqBa, SCstqBa-D, SCstqBa-T, SCstqBa-B respectively.

The alphabet of the languages of SCstqBa, SCstqBa-D, SCstqBa-T, SCstqBa-B consists of

**Fig. 5.** Hasse diagram (stqBa-T).

- variables r, s, t, ...
- constants $\bot$ and $\top$
- Unary logical connectives $\neg$ and *i*.
- Binary logical connectives $\wedge$ and $\vee$.
- parentheses (,).

*c* is a definable connective, defined as $c\gamma = \neg i \neg \gamma$.
In usual way, formulae are formed and denoted by $\alpha, \beta, \gamma, \delta, \ldots$ etc.
**Axioms and rules for SCstqBa**: Axioms and rules of the logic system SCstqBa are as follows:

$$\begin{array}{llll}
Ax1 & \gamma \Rightarrow \neg\neg\gamma & Ax2 & \neg\neg\gamma \Rightarrow \gamma \\
\text{Cut} & \dfrac{A \Rightarrow \gamma, B \qquad C, \gamma \Rightarrow D}{A, C \Rightarrow B, D} & \text{Rule } \neg & \dfrac{A \Rightarrow B}{\neg B \Rightarrow \neg A} \\
\text{LW} & \dfrac{A \Rightarrow B}{A, \gamma \Rightarrow B} & \text{RW} & \dfrac{A \Rightarrow B}{A \Rightarrow \gamma, B} \\
\text{LC} & \dfrac{A, \gamma, \gamma \Rightarrow B}{A, \gamma \Rightarrow B} & \text{RC} & \dfrac{A \Rightarrow \gamma, \gamma, B}{A \Rightarrow \gamma, B} \\
\text{L}\vee & \dfrac{A, \gamma \Rightarrow B \qquad C, \delta \Rightarrow D}{A, C, \gamma \vee \delta \Rightarrow B, D} & \text{R}\vee & \dfrac{A \Rightarrow \gamma, \delta, B}{A \Rightarrow \gamma \vee \delta, B} \\
\text{L}\wedge & \dfrac{A, \gamma, \delta \Rightarrow B}{A, \gamma \wedge \delta \Rightarrow B} & \text{R}\wedge & \dfrac{A \Rightarrow \gamma, B \qquad C \Rightarrow \delta, D}{A, C \Rightarrow \gamma \wedge \delta, B, D} \\
L\bot & A, \bot \Rightarrow B & R\top & A \Rightarrow \top, B.
\end{array}$$

The above axioms and rules have been used to present the sequent calculus of qBa [12,8]. For SCstqBa, additionally, one axiom and two rules are needed. These are:

$Ax3 \quad i\gamma, i\delta \Rightarrow i(\gamma \wedge \delta)$

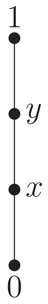


**Fig. 6.** Hasse diagram (stqBa-B).

Rule $i$ $\frac{\gamma\Rightarrow\delta}{i\gamma\Rightarrow i\delta}$ $(\mathrm{R}i)^{\mathrm{r}}$ $\frac{\Rightarrow\gamma}{\Rightarrow i\gamma}$.

In Rule $\neg$, $\neg A$ means $\neg\alpha_1, \neg\alpha_2, \cdots, \neg\alpha_n$when $A$ is $\alpha_1, \alpha_2, \cdots, \alpha_n$. Moreover, here, $Ax$ stands for axiom and the rules LW, RW, LC, RC, L$\vee$, R$\vee$, L$\wedge$, R$\wedge$, $R(i)^r$ stand for left weakening, right weakening, left contraction, right contraction (these are called structural rules), left rule for $\vee$, right rule for $\vee$, left rule for $\wedge$, right rule for $\wedge$, restricted right rule for i (these are called rules for connectives) respectively. Besides, $L\perp$ and $R\top$ are two rules for constants respectively called rule for bottom ($\perp$ to be placed at the left side of the sequent) and rule for top ($\top$ to be placed at the right side of the sequent).

Further, it is to be noted that no exchange rules viz. LE (left exchange rule) and RE (right exchange rule):

LE $\frac{A,\gamma,\delta,B\Rightarrow C}{A,\delta,\gamma,B\Rightarrow C}$ RE $\frac{A\Rightarrow B,\gamma,\delta,C}{A\Rightarrow B,\delta,\gamma,C}$ are not required in these sequent calculi as $A, B$ are taken multisets to define a sequent $A \Rightarrow B$. In a multiset, order of occurrence of elements does not matter e.g. $\{\alpha, \beta, \alpha, \gamma\}$ is the same as $\{\beta, \gamma, \alpha, \alpha\}$.

**Axioms and rules for SCstqBa-D**: All axioms and rules of the logic system SCstqBa along with one new axiom

$Ax4$ $i\gamma \Rightarrow c\gamma$.

**Axioms and rules for SCstqBa-T**: All axioms and rules of the logic system SCstqBa along with one new rule:

$(\mathrm{L}i)^{\mathrm{r}}$ $\frac{\gamma\Rightarrow\delta}{i\gamma\Rightarrow\delta}$

**Axioms and rules for SCstqBa-B**: All axioms and rules of the logic system SCstqBa-T along with one new axiom

$Ax5$ $ci\gamma \Rightarrow \gamma$.

**Definition 3.1.** A model of SCstqBa/SCstqBa-D/SCstqBa-T/SCstqBa-B is $\langle \mathbb{U}, v \rangle$ where $\mathbb{U} = \langle U, \wedge, \vee, \neg, I, 0, 1 \rangle$ is a stqBa/stqBa-D/stqBa-T/stqBa-B algebra and $v$ is a valuation function defined as $v(\perp) = 0$, $v(\top) = 1$ and for each atomic wff $p$ of the language of SCstqBa/SCstqBa-D/SCstqBa-T/ SCstqBa-B, $v(p) \in U$.

**Remark 3.1.** Any valuation function $v$ can be extended to arbitrary formulae as follows $v(\gamma \wedge \delta) = v(\gamma) \wedge v(\delta)$, $v(\gamma \vee \delta) = v(\gamma) \vee v(\delta)$, $v(\neg\gamma) = \neg v(\gamma)$, $v(i\gamma) = Iv(\gamma)$, $v(c\gamma) = Cv(\gamma)$ where $Cx = \neg I \neg x$.

**Definition 3.2.** A sequent $A \Rightarrow B$, where both $A, B$ are non empty multisets of wffs is said to be valid in a model $\langle \mathbb{U}, v \rangle$ of SCstqBa/SCstqBa-D/SCstqBa-T/SCstqBa-B if and only if $v(A) \leqslant v(B)$, where $v(A)$ means $v(\alpha) \wedge v(\beta) \wedge v(\gamma) \ldots \wedge v(\delta)$ when $A$ is $\alpha, \beta, \gamma \ldots, \delta$ and $v(B)$ means $v(\alpha) \vee v(\beta) \vee v(\gamma) \ldots \vee v(\delta)$ when $B$ is $\alpha, \beta, \gamma \ldots, \delta$. When $A$ is $\emptyset$, i.e., the sequent is of the form $\Rightarrow B$, it is said to be valid if and only if $v(B) = 1$ and when $B$ is $\emptyset$, i.e., the sequent is of the form $A \Rightarrow$, it is said to be valid if and only if $v(A) = 0$, $v(B)$ and $v(A)$ are defined in the same way as stated above.

**Theorem 3.1.** Systems SCstqB, SCstqBa-D, SCstqBa-T and SCstqBa-B are sound and complete with respect to the class of all algebras stqBa, stqBaD, stqBaT, stqBaB respectively.

**Proof.** (Outline) Here, we consider the logic system SCstqBa to prove the result. Proof of remaining systems are similar. To prove soundness, it is to be established that all axioms and rules in the logic system SCstqBa are valid. Mathematical induction is to be applied on n, the depth of the derivation of the sequent, to prove its validity.

For completeness, let us assume that $H$ is the set of all wffs of the logic system SCstqBa. To obtain the Lindenbaum-Tarski algebra for the logic system SCstqBa, a relation $R$ on $H$ is defined by $\gamma R \delta$ if and only if $\gamma \Rightarrow \delta$ and $\delta \Rightarrow \gamma$ are derivable sequents, where $\gamma$ and $\delta$ are any two well formed formulae. Then, $R$ is an equivalence relation due to the presence of $Ax1, Ax2$ and Cut. The Lindenbaum- Tarski algebra for the logic system SCstqBa with connectives $\wedge, \vee, \neg, i, c$ is $\langle H/R, \wedge, \vee, \neg, I, C \rangle$, where the operations $\wedge, \vee, \neg, I, C$are defined as $[\gamma] \wedge [\delta] = [\gamma \wedge \delta], [\gamma] \vee [\delta] = [\gamma \vee \delta], \neg[\gamma] = [\neg\gamma], I[\gamma] = [i\gamma], C[\gamma] = [c\gamma]$. The partial order relation $\leqslant, [\gamma] \leqslant [\delta]$ if and only if $\gamma \Rightarrow \delta$ is a derivable sequent, yields $\langle H/R, \wedge, \vee, \neg, I, 0, 1 \rangle$ as a stqBa ($\leqslant$being the lattice order) where $0 = [\perp], 1 = [\top]$. Now, we consider the canonical valuation $v$, i.e., $v(p) = [p]$, for all atomic wffs $p \in H$. It can be extended over $H$ as $v(\gamma) = [\gamma]$, for all $\gamma \in H$. Then, $\langle \mathbb{U}, v \rangle$ is a model of SCstqBa where $\mathbb{U} = \langle H/R, \wedge, \vee, \neg, I, 0, 1 \rangle$. Let $A \Rightarrow B$ be a valid sequent in every model of SCstqBa. Then, it is valid in the model $\langle \mathbb{U}, v \rangle$ and consequently $v(A) \leqslant v(B)$. Using axioms and rules of SCstqBa, it can be shown that $A \Rightarrow B$ is derivable in the logic system SCstqBa. So, completeness is established.

## 4. Rough set models of some algebras

In this section, proper rough set models of some of the algebras presented in Section 2 will be developed. For this, as mentioned in the introduction, a pair of lower–upper approximations is needed which must be dual with respect to the quasi-complementation. The notion of *quasi – complementation* [7] has been discussed in the introduction. Proposition 4.1 below describes some of its properties.

**Proposition 4.1.** [7] Let $g : U \to U$ be an involution, i.e., $g(g(u)) = u$, for all $u \in U$. The following results hold.

1. $g$ is a bijective mapping on $U$.
2. $g(g(P)) = P$, for all $P \subseteq U$.

3. $g(P \cup Q) = g(P) \cup g(Q)$, for all $P, Q \subseteq U$.
4. $g(P \cap Q) = g(P) \cap g(Q)$, for all $P, Q \subseteq U$.
5. $\sim P = g(P^c)$, for all $P \subseteq U$ ($\sim P = U - g(P) = (g(P))^c$ is already defined in the introduction).
6. $\sim\sim P = P$, for all $P \subseteq U$.
7. $\sim (P \cap Q) = \sim P \cup \sim Q$, for all $P, Q \subseteq U$.
8. $\sim (P \cup Q) = \sim P \cap \sim Q$, for all $P, Q \subseteq U$.

Our recent paper [11] provides a pair of lower–upper approximations which are dual with respect to the quasi-complementation. In this paper, a $g$-approximation space $\langle U, R^g \rangle$ has been defined as follows:

Let $\langle U, R \rangle$ be a generalised approximation space and $g : U \rightarrow U$ be an involution. A binary relation $R^g$ on $U$ has been defined as:

$$\text{for any two elements } u \text{ and } v \in U, uR^g v \text{ if and only if } g(u)Rg(v). \tag{6}$$

That is, two elements $u, v \in U$ are related with respect to a new relation $R^g$ if and only if their $g$-images are related in the relation $R$.

$\langle U, R^g \rangle$ is called a $g$-generalised approximation space or simply, a $g$-approximation space.

As $g$ is an involution on $U, R$ can be redefined from $R^g$ as follows:

$$\text{for any two elements } u \text{ and } v \in U, \ uRv \text{ if and only if } g(u)R^g g(v). \tag{7}$$

Similarly, it says that two elements $u, v \in U$ will be related in the relation $R$ if and only if their $g$-images are so in the relation $R^g$.

In general, there is no subset inclusion relation between $R$ and $R^g$. However, the following results show how they are related depending upon $g$.

**Proposition 4.2.** [11] The following statements are equivalent in a $g$-approximation space $\langle U, R^g \rangle$.

1. $R^g = R$.
2. $uRv$ implies $g(u)Rg(v), \forall u, v \in U$.
3. $g(u)Rg(v)$ implies $uRv, \forall u, v \in U$.
4. $uR^g v$ implies $g(u)R^g g(v), \forall u, v \in U$.
5. $g(u)R^g g(v)$ implies $uR^g v, \forall u, v \in U$.
6. $R \subseteq R^g$.
7. $R^g \subseteq R$.

Let $R_u = \{v \in U : uRv\}$ and $R^g_u = \{v \in U : uR^g v\}$. As $g$ is an involution, it is obvious that $R_{g(g(u))} = R_u$ and $R^g_{g(g(u))} = R^g_u$, for all $u \in U$. But, there is no subset inclusion relation amongst $R_u, R_{g(u)}, R^g_u$ and $R^g_{g(u)}$ in general. The following results show how they are linked depending upon $R$ and $g$.

**Proposition 4.3.** [11] The following statements are equivalent in a $g$-approximation space $\langle U, R^g \rangle$.

1. $R^g_u = R^g_{g(u)} \left(R_u = R_{g(u)}\right), \forall u \in U$.
2. $uR^g v(uRv)$implies $g(u)R^g v(g(u)Rv), \forall u, v \in U$.
3. $g(u)R^g v(g(u)Rv)$implies $uR^g v(uRv), \forall u, v \in U$.
4. $R^g_u \subseteq R^g_{g(u)} \left(R_u \subseteq R_{g(u)}\right), \forall u \in U$.
5. $R^g_{g(u)} \subseteq R^g_u \left(R_{g(u)} \subseteq R_u\right), \forall u \in U$.
6. $R_u = R_{g(u)} \left(R^g_u = R^g_{g(u)}\right), \forall u \in U$.

**Proposition 4.4.** [11] In a $g$-approximation space $\langle U, R^g \rangle, R_u = g\left(R^g_{g(u)}\right)$and $R^g_u = g\left(R_{g(u)}\right)$, for all $u \in U$.

**Proposition 4.5.** [11] In a $g$-approximation space $\langle U, R^g \rangle$ the following results hold.

1. $R^g$ is reflexive if and only if $R$ is reflexive.
2. $R^g$ is symmetric if and only if $R$ is symmetric.
3. $R^g$ is transitive if and only if $R$ is transitive.

4. $R^g$ is serial if and only if $R$ is serial.

**Proposition 4.6.** [11] If $R^g(R)$ is reflexive and transitive and $R^g_u = R^g_{g(u)}\left(R_u = R_{g(u)}\right)$, for all $u \in U$ then $R^g = R$.

We now present the lower–upper approximations that have been defined in [11].

Let $\langle U, R^g \rangle$ be a $g$-approximation space. The $g$-lower approximation and the $g$- upper approximation $_g, ^g : 2^U \to 2^U$ are defined as: for any $P \in 2^U$,

$$\underline{P}_g = \{u \in U : R^g_u \subseteq P\} \tag{8}$$

and

$$\overline{P}^g = \left\{u \in U : R^g_{g(u)} \cap g(P) \neq \emptyset\right\}. \tag{9}$$

It has been shown that $g$-lower approximation $\underline{P}_g$ and $g$-upper approximation $\overline{P}^g$ are dual to each other with respect to the quasi-complementation $\sim$ defined through g, i.e., $(\underline{\sim P}_g) = \sim (\overline{P}^g)$ and $\overline{(\sim P)}^g = \sim (\underline{P}_g)$. Moreover, we have proved that $\underline{P}_g$ and $\overline{P}^g$ are respectively Pawlakian lower approximation of $P$ in the approximation space $\langle U, R^g \rangle$ and Pawlakian upper approximation of $P$ in the approximation space $\langle U, R \rangle$, i.e., $\underline{P}_g = \underline{P}_{R^g} = \left\{x \in U : R^g_x \subseteq P\right\}$ and $\overline{P}^g = \overline{P}^R = \{x \in U : R_x \cap P \neq \emptyset\}$.

It has been mentioned earlier that $\underline{P}_g$ and $\overline{P}^g$ are dual approximations with respect to the quasi-complementation. But $\underline{P}_R$ and $\overline{P}^R$ are not so. In fact, they are dual with respect to the set theoretic complementation. A necessary and sufficient condition is presented below so that $\underline{P}_R$ and $\overline{P}^R$ are dual approximations with respect to the quasi-complementation.

**Theorem 4.1.** [11] Let $\langle U, R \rangle$ be a generalised approximation space and $g$ be an involution on $U$. Then for any $P \subseteq U$, $\underline{P}_R$ and $\overline{P}^R$ are dual approximations with respect to the quasi-complementation defined through $g$ if and only if $R = R^g$.

**Remark 4.1.** From Theorem 4.1 one can say, in other words, that for a fixed involution $g$ on $U$, it is possible to find a collection of relations $\{R : R = R^g\}$ so that Pawlakian lower- upper approximations $\underline{P}_R$ and $\overline{P}^R$ are dual with respect to the quasi-complementation as well.

However, when $R \neq R^g$, $\langle \underline{P}_g, \overline{P}^g \rangle \neq \langle \underline{P}_R, \overline{P}^R \rangle$ and $\langle \underline{P}_g, \overline{P}^g \rangle \neq \langle \underline{P}_{R^g}, \overline{P}^{R^g} \rangle$, yet the following results hold.

**Proposition 4.7.** [11] In a $g$-approximation space $\langle U, R^g \rangle$, the following results hold.

1. $\underline{U}_g = U$ and $\overline{\emptyset}^g = \emptyset$.
2. If $P \subseteq Q \subseteq U$ then $\underline{P}_g \subseteq \underline{Q}_g$ and $\overline{P}^g \subseteq \overline{Q}^g$.
3. $\underline{P \cap Q}_g = \underline{P}_g \cap \underline{Q}_g$ and $\overline{P \cup Q}^g = \overline{P}^g \cup \overline{Q}^g$, for all $P, Q \subseteq U$.

The counterpart of the modal axiom K (in quasi-Boolean base) in the form $\underline{\sim P \cup Q}_g \subseteq \sim (\underline{P}_g) \cup \underline{Q}_g$ does not hold in general. The following example shows this.

**Example 4.1.** Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $g : U \to U$ be an involution defined by $g(u_1) = u_4, g(u_2) = u_6, g(u_3) = u_3, g(u_4) = u_1, g(u_5) = u_5, g(u_6) = u_2$. Let $R = \{(u_1, u_1), (u_1, u_4), (u_4, u_1), (u_3, u_6)\}$ be a relation on $U$. Then, $R^g = \{(u_4, u_4), (u_4, u_1), (u_1, u_4), (u_3, u_2)\}$. Let $P = \{u_2, u_3, u_4\}$ and $Q = \{u_5, u_6\}$. Then, $\underline{\sim P \cup Q}_g = \{u_1, u_2, u_3, u_5, u_6\}$and $\sim (\underline{P}_g) \cup \underline{Q}_g = \{u_1, u_2, u_5, u_6\}$. Thus, $\underline{\sim P \cup Q}_g \nsubseteq \sim (\underline{P}_g) \cup \underline{Q}_g$.

**Proposition 4.8.** A sufficient condition so that $\underline{\sim P \cup Q}_g \subseteq \sim (\underline{P}_g) \cup \underline{Q}_g$ holds, for all $P, Q \subseteq U$, is that $R = R^g$.

**Proof.** Let $u \in \underline{\sim P \cup Q}_g$. Then, $R^g_u \subseteq (g(P))^c \cup Q$. Two possible cases are

1. $u \in g(\underline{P}_g)$
2. $u \notin g(\underline{P}_g)$.

For the second case, it is obvious that $u \in \sim (\underline{P}_g) \cup \underline{Q}_g$. For the first case, $u = g(v)$ where $v \in \underline{P}_g$. Then, $R^g_v \subseteq P$ and hence $g(R^g_v) \subseteq g(P)$, i.e., $R_u \subseteq g(P)$ [by Proposition 4.4 and $u = g(v)$]. As, $R = R^g$ so, $R_u = R^g_u$ and therefore $R^g_u \cap (g(P))^c = \emptyset$. Then, $R^g_u \subseteq Q$ [as $R^g_u \subseteq (g(P))^c \cup Q$] and therefore $u \in \underline{Q}_g$. Hence the result follows.

**Remark 4.2.**

1. When $R = R^g, \underline{P}_g$ and $\overline{P}^g$ become $\underline{P}_R$ and $\overline{P}^R$ respectively. Then $\underline{\sim P \cup Q}_g \subseteq \sim (\underline{P}_g) \cup \underline{Q}_g$ turns into $\underline{\sim P \cup Q}_R \subseteq \sim (\underline{P}_R) \cup \underline{Q}_R$ which is not identical with $\underline{P^c \cup Q}_R \subseteq (\underline{P}_R)^c \cup \underline{Q}_R$ (the counterpart of the modal axiom K in Boolean base)
2. Whether $R = R^g$ is a necessary condition or not for holding $\underline{\sim P \cup Q}_g \subseteq \sim (\underline{P}_g) \cup \underline{Q}_g$ is unsolved.

When $R$ is a serial relation on $U$, Pawlakian lower–upper approximations $\underline{P}_R$ and $\overline{P}^R$ satisfy the counterpart of the modal axiom D: $\underline{P}_R \subseteq \overline{P}^R$ [13,10]. But, the following Example 4.2 shows that $\underline{P}_g \subseteq \overline{P}^g$ does not hold for a serial relation $R^g$.

**Example 4.2.** $U$ and $g$ are the same as stated in Example 4.1. Let $R = \{(u_1, u_2), (u_2, u_1), (u_3, u_1), (u_4, u_6), (u_5, u_5), (u_6, u_2)\}$ be a serial relation on $U$. Then, $R^g = \{(u_4, u_6), (u_6, u_4), (u_3, u_4), (u_1, u_2), (u_5, u_5), (u_2, u_6)\}$ is a serial relation on $U$. Let $P = \{u_2, u_4, u_5\}$. Then, $\underline{P}_g = \{u_1, u_3, u_5, u_6\}$and $\overline{P}^g = \{u_1, u_5, u_6\}$. Thus, $\underline{P}_g \not\subseteq \overline{P}^g$ .

A necessary and sufficient condition is obtained below so that $\underline{P}_g \subseteq \overline{P}^g$ holds in a g-approximation space $\langle U, R^g \rangle$.

**Theorem 4.2.** In a $g$-approximation space $\langle U, R^g \rangle, \underline{P}_g \subseteq \overline{P}^g$ holds for all $P \subseteq U$ if and only if $R^g_u \cap R_u \neq \emptyset$, for all $u \in U$.

**Proof.** Let us assume that $\underline{P}_g \subseteq \overline{P}^g$ holds for all $P \subseteq U$. Let $u \in U$. Then, particularly, $\underline{R^g_u}_g \subseteq \overline{R^g_u}^g$ holds. This gives, $u \in \left\{ v \in U : R_v \cap R^g_u \neq \emptyset \right\}$ [as $u \in \underline{R^g_u}_g$] and hence $R^g_u \cap R_u \neq \emptyset$. Conversely, let $R^g_u \cap R_u \neq \emptyset$, for all $u \in U$. Let $u \in \underline{P}_g$. This implies $R^g_u \cap R_u \subseteq R_u \cap P$ and therefore $R_u \cap P \neq \emptyset$ [as $R^g_u \cap R_u \neq \emptyset$]. Hence, $u \in \overline{P}^R = \overline{P}^g$.

**Remark 4.3.** $R^g_u \cap R_u \neq \emptyset$, for all $u \in U$ implies that $R^g(R)$ is a serial relation on $U$. But the converse is not true, i.e., there exists a serial relation $R^g(R)$ so that $R^g_u \cap R_u = \emptyset$, for some $u \in U$. In Example 4.2, $R^g_2 \cap R_2 = \emptyset$. Thus, the condition in Theorem 4.2 is stronger than a serial relation. By Example 4.2, it is also noted that $R^g_u \cap R_u \neq \emptyset$, for all $u \in U$ does not imply $R = R^g$.

**Proposition 4.9.** [11] If $R^g$ is reflexive in a $g$-approximation space $\langle U, R^g \rangle$, the following results hold.

1. $\overline{U}^g = U$ and $\underline{\emptyset}_g = \emptyset$.
2. $\underline{P}_g \subseteq P \subseteq \overline{P}^g$, for all $P \subseteq U$.

It is known to us that Pawlakian lower–upper approximations $\underline{P}_R$ and $\overline{P}^R$ satisfy the counterpart of the modal axiom B: $\overline{(\underline{P}_R)}^R \subseteq P$, for all $P \subseteq U$, when $R$ is a symmetric relation on $U$. But, $\overline{(\underline{P}_g)}^g \subseteq P$does not hold even for an equivalence relation $R^g$ on $U$ as shown in Example 4.3 below.

**Example 4.3.** $U$ and $g$ are the same as mentioned in Example 4.1. Let $R$ be an equivalence relation on $U$ which partitions the set $U$ into the subsets $\{u_2, u_3\}, \{u_4\}, \{u_1, u_5\}, \{u_6\}$ of $U$. Then, the equivalence relation $R^g$ partitions the set $U$ into the subsets $\{u_3, u_6\}, \{u_1\}, \{u_4, u_5\}, \{u_2\}$ of $U$. Let $P = \{u_1, u_3, u_6\}$. Then, $\underline{P}_g = \{u_1, u_3, u_6\}$ and $\overline{(\underline{P}_g)}^g = \{u_1, u_2, u_3, u_5, u_6\}$ and therefore $\overline{(\underline{P}_g)}^g \not\subseteq P$.

A necessary and sufficient condition is presented below so that the counterpart of the modal axiom B holds in a g-approximation space $\langle U, R^g \rangle$.

**Theorem 4.3.** Let $R^g$ be a symmetric relation in a $g$-approximation space $\langle U, R^g \rangle$. Then for any subset $P$ of $U$, $\overline{(\underline{P}_g)}^g \subseteq P$ holds if and only if $R^g = R$.

**Proof.** Let $R^g = R$. Then, $\langle \underline{P}_g, \overline{P}^g \rangle = \langle \underline{P}_R, \overline{P}^R \rangle$ and consequently for any subset $P$ of $U$, $\overline{(\underline{P}_g)}^g \subseteq P$ holds. Conversely, let $\overline{(\underline{P}_g)}^g \subseteq P$ hold, for any subset $P$ of $U$. We shall show that $R \subseteq R^g$. If $R = \emptyset$ then $R^g = \emptyset$ and hence it is done. Let $uRv$ and $P = R^g_v$. Then, $\overline{\left(\underline{R^g_v}_g\right)}^g \subseteq R^g_v$ i.e., $\left\{ z \in U : R_z \cap \underline{R^g_v}_g \neq \emptyset \right\} \subseteq R^g_v$. Now, $\underline{R^g_v}_g = \{ w \in U : R^g_w \subseteq R^g_v \}$ and hence $v \in \underline{R^g_v}_g$. As $v \in R_u$[since $uRv$ is assumed], $R_u \cap \underline{R^g_v}_g \neq \emptyset$ and therefore $u \in R^g_v$. This gives $uR^g v$ as $R^g$ is symmetric. Thus, $R \subseteq R^g$. Using Remark 4.2, $R^g = R$.

**Remark 4.4.** By the above theorem, it is clear that the counterpart of the modal axiom B is possible with respect to $g$-lower and $g$-upper approximations only when $R^g = R$. Indeed, in that case, all the properties of lower/upper approximations with respect to $R$ as well as $R^g$ coincide. Yet, there remains one significant point. The complementation and the quasi-complementation do not coincide still the approximation operators are dual with respect to both of them.

**Proposition 4.10.** [11] If $R^g$ is transitive in a $g$-approximation space $\langle U, R^g \rangle$ then for any subset $P$ of $U$, $\underline{P}_g \subseteq \underline{(\underline{P}_g)}_g$ and $\overline{(\overline{P}^g)}^g \subseteq \overline{P}^g$ hold.

It has been shown [11] that the counterpart of the modal axiom $S_5 : \overline{(\underline{P}_g)}^g \subseteq \underline{P}_g$ may not hold for an equivalence relation $R^g$ in a $g$-approximation space $\langle U, R^g \rangle$. The following Theorem 4.4 has been established in order to obtain $\overline{(\underline{P}_g)}^g \subseteq \underline{P}_g$ in a $g$-approximation space $\langle U, R^g \rangle$.

**Theorem 4.4.** [11] Let $R^g$ be an equivalence relation in a $g$-approximation space $\langle U, R^g \rangle$. Then for any subset $P$ of $U$, $\overline{(\underline{P}_g)}^g \subseteq \underline{P}_g$ holds if and only if $R^g = R$.

The following example is considered to show that the counterpart of IP1: $\sim(\underline{P}_g) \cup \underline{P}_g = U$ may not hold even for an equivalence relation $R^g$ with $R^g = R$.

**Example 4.4.** $U$ and $g$ are the same as stated in Example 4.1. Let $R = \{(u_1, u_1), (u_2, u_2), (u_3, u_3), (u_4, u_4), (u_5, u_5), (u_6, u_6), (u_1, u_4), (u_4, u_1)\}$. Then $R$ is an equivalence relation on $U$ with $R = R^g$. Let $P = \{u_1, u_2, u_3\}$. Then, $\sim(\underline{P}_g) \cup \underline{P}_g = \{u_1, u_2, u_3, u_4, u_5\} \neq U$.

We now state a necessary and sufficient condition so that $\sim(\underline{P}_g) \cup \underline{P}_g = U$ holds, for all $P \subseteq U$.

**Theorem 4.5.** Let $R^g$ $(R)$ be an arbitrary relation in a $g$-approximation space $\langle U, R^g \rangle$ (generalized approximation space $\langle U, R \rangle$). Then for any subset $P$ of $U$, $\sim(\underline{P}_g) \cup \underline{P}_g = U$ $(\sim(\underline{P}_R) \cup \underline{P}_R = U)$ holds if and only if $R^g_u = R^g_{g(u)}$ $(R_u = R_{g(u)})$, for all $u \in U$.

**Proof.** Let $R^g_u = R^g_{g(u)}$, for all $u \in U$. Let $P$ be any subset of $U$ and $v$ be any element of $U$. If $v \in \sim(\underline{P}_g)$, it is done. So, let $v \notin \sim(\underline{P}_g)$. Then, $v \notin U - \{g(u) : R^g_u \subseteq P\}$ and hence $v \in \{g(u) : R^g_u \subseteq P\}$. Then, $v = g(t)$ where $R^g_t \subseteq P$. As $R^g_t = R^g_{g(t)}$ [by the hypothesis], it follows that $v \in \underline{P}_g$ [since $g(t) = v$]. Thus, $\sim(\underline{P}_g) \cup \underline{P}_g = U$, for any subset $P$ of $U$. Conversely, let $\sim(\underline{P}_g) \cup \underline{P}_g = U$, for any subset $P$ of $U$. Let $u \in U$. It is to be shown that $R^g_u = R^g_{g(u)}$. Let $P = R^g_{g(u)}$. Then, $\underline{P}_g = \left\{v : R^g_v \subseteq R^g_{g(u)}\right\}$. We now claim that $u \in \underline{P}_g$. If not, $u \in \sim(\underline{P}_g)$ as $\sim(\underline{P}_g) \cup \underline{P}_g = U$. Then, $u \notin \left\{g(v) : R^g_v \subseteq R^g_{g(u)}\right\}$. As $g$ is bijective on $U$, let $u = g(z)$, and then $R^g_z \not\subseteq R^g_{g(u)}$, i.e., $R^g_{g(u)} \not\subseteq R^g_{g(u)}$ [as $u = g(z)$ implies $z = g(u)$], which is a contradiction. Thus, $u \in \underline{P}_g = \left\{v : R^g_v \subseteq R^g_{g(u)}\right\}$and hence$R^g_u \subseteq R^g_{g(u)}$. Using Remark 4.3, $R^g_u = R^g_{g(u)}$.

**Note 4.1.** As $R^g_u = R^g_{g(u)}$ implies and implied by $R_u = R_{g(u)}$ [by Proposition 4.4], both $\sim(\underline{P}_g) \cup \underline{P}_g = U$ and $\sim(\underline{P}_R) \cup \underline{P}_R = U$ in Theorem 4.5 hold good for any one of the conditions $R^g_u = R^g_{g(u)}$ and $R_u = R_{g(u)}$.

In order to view some of the results of this section at a glance we refer to Table 1.

### 4.1. Rough set models for stqBa, System0 algebra, stqBa-D, stqBa-T, stqBa-B, tqBa, tqBa5 and IA1

In respect of the results of algebras developed in Section 2 and the properties of $\underline{P}_g$ and $\overline{P}^g$ introduced in this section, rough set models for the algebras stqBa, System0 algebra, stqBa-D, stqBa-T, stqBa-B, tqBa, tqBa5 and IA1 are presented below.

**Rough Set model for a stqBa**: Let $\langle U, R^g \rangle$ be a $g$-approximation space. Then, $\left\langle 2^U, \cap, \cup, \sim, \emptyset, U \right\rangle$ is a qBa, where $\sim P = (g(P))^c$, for all $P \in 2^U$ [by Proposition 4.1]. We now define $IP$, for all $P \subseteq U$ as $IP = \underline{P}_g$. Then by Proposition 4.1 and Proposition 4.7, $\left\langle 2^U, \cap, \cup, \sim, I, \emptyset, U \right\rangle$ is a stqBa.

**Remark 4.5.** By Proposition 4.1, Proposition 4.7 and Definition 2.2, the above model of stqBa is also a model for System0 algebra.

**Rough Set model for a stqBa-D**: Let $R^g$ be a relation on $U$ so that $R^g_u \cap R_u \neq \emptyset$, for all $u \in U$. Then, by Proposition 4.1, Proposition 4.7 and Theorem 4.2, $\left\langle 2^U, \cap, \cup, \sim, I, \emptyset, U \right\rangle$ is a stqBa-D.

**Rough Set model for a stqBa-T**: For a reflexive relation $R^g$ on $U$, by Proposition 4.1, Proposition 4.7 and Proposition 4.9, $\left\langle 2^U, \cap, \cup, \sim, I, \emptyset, U \right\rangle$ is a stqBa-T.

**Rough Set model for a stqBa-B**: For a reflexive and symmetric relation $R^g$ on $U$ with $R^g = R$, by Proposition 4.1, Proposition 4.7, Proposition 4.9 and Theorem 4.3, $\left\langle 2^U, \cap, \cup, \sim, I, \emptyset, U \right\rangle$ is a stqBa-B.

**Remark 4.6.** By Proposition 4.8, the algebraic counterpart of the modal axiom K holds in the above model of stqBa-B as $R^g = R$. Thus, it is also a model for the abstract algebra stqBa-B along with modal axiom K in the form $I(\neg x \vee y) \leqslant \neg Ix \vee Iy$.

**Table 1**
Some results on the two lower–upper approximations

| Nature of $R$ | Result |
|---|---|
| $R$ is arbitrary but $R \neq R^g$ | (1) $\underline{P}_g$ and$\overline{P}^g$ are dual with respect to the quasi-complementation.<br>(2) $\overline{P}^g = \overline{P}^R$.<br>(3) $\underline{P^c \cup Q}_R \subseteq (\underline{P}_R)^c \cup \underline{Q}_R$, i.e, the modal axiom K in Boolean base holds.<br>(4) $\underline{\sim P \cup Q}_g \not\subseteq \sim (\underline{P}_g) \cup \underline{Q}_g$, i.e, the modal axiom K in quasi-Boolean base does not hold.<br>(5) $\underline{P}_g \neq \underline{P}_R$.<br>(6) $\underline{P}_R$ and $\overline{P}^R$ are not dual with respect to the quasi-complementation. |
| $R$ is arbitrary but $R = R^g$ | (1) $\underline{P}_g = \underline{P}_R$.<br>(2)$\overline{P}^g = \overline{P}^R$.<br>(3) $\underline{\sim P \cup Q}_g \subseteq \sim (\underline{P}_g) \cup \underline{Q}_g$, i.e, the modal axiom K in the quasi-Boolean base holds.<br>(4) $\underline{P}_R$ and $\overline{P}^R$ are always dual with respect to both the quasi-complementation and complementation. |
| $R$ is a serial relation with $R_u \cap R^g_u = \emptyset$, for at least one $u \in U$ | (1)$\underline{P}_R \subseteq \overline{P}^R$ holds for all $P \subseteq U$.<br>(2) $\underline{P}_g \subseteq \overline{P}^g$ does not hold for at least one $P \subseteq U$. |
| $R$ is a (serial) relation with $R_u \cap R^g_u \neq \emptyset$, for all $u \in U$ | (1)$\underline{P}_R \subseteq \overline{P}^R$ holds for all $P \subseteq U$.<br>(2) $\underline{P}_g \subseteq \overline{P}^g$ holds for all $P \subseteq U$. |
| $R$ is reflexive but $R \neq R^g$ | (1) $\underline{P}_g \subseteq P \subseteq \overline{P}^g = \overline{P}^R$.<br>(2) $\underline{P}_R$ and $\overline{P}^R$ are not dual with respect to the quasi-complementation.<br>(3)$\underline{P}_g, \underline{P}_R \subseteq P$ but there is no fixed subset inclusion relation between $\underline{P}_R$ and $\underline{P}_g$. |
| $R$ is symmetric but $R \neq R^g$ | (1) $\overline{(\underline{P}_R)}^R \subseteq P$<br>(2) $\overline{(\underline{P}_g)}^g \not\subseteq P$ |
| $R$ is transitive | (1) $\underline{P}_R \subseteq \underline{(\underline{P}_R)}_R$ and $\overline{(\overline{P}^R)}^R \subseteq \overline{P}^R$.<br>(2) $\underline{P}_g \subseteq \underline{(\underline{P}_g)}_g$ and $\overline{(\overline{P}^g)}^g \subseteq \overline{P}^g$. |
| $R$ is equivalence but $R \neq R^g$ | (1) $\overline{(\underline{P}_R)}^R \subseteq \underline{P}_R$.<br>(2) $\overline{(\underline{P}_g)}^g \not\subseteq \underline{P}_g$.<br>(3) $\underline{P}_g \subseteq P \subseteq \overline{P}^g = \overline{P}^R$.<br>(4) $\underline{P}_R$ and $\overline{P}^R$ are not dual with respect to the quasi-complementation.<br>(5)$\underline{P}_g, \underline{P}_R \subseteq P$ but there is no fixed subset inclusion relation between $\underline{P}_g$ and $\underline{P}_R$. |
| $R$ is any relation with $R_u = R_{g(u)}$, for all $u$ | (1) $\sim (\underline{P}_R) \cup \underline{P}_R = U$.<br>(2) $\sim (\underline{P}_g) \cup \underline{P}_g = U$. |

**Rough Set model for a tqBa**: For any reflexive and transitive relation $R^g$ on $U$, by Proposition 4.1, Proposition 4.7, Proposition 4.9 and Proposition 4.10, $\langle 2^U, \cap, \cup, \sim, I, \emptyset, U \rangle$ is a tqBa.

**Rough Set model for a tqBa5**: For any equivalence relation $R^g$ on $U$ with $R^g = R$, by Proposition 4.1, Propositions 4.7, Proposition 4.9, Proposition 4.10 and Theorem 4.4, $\langle 2^U, \cap, \cup, \sim, I, C, \emptyset, U \rangle$ is a tqBa5, where $IP = \underline{P}_g = \underline{P}_R$ and $CP = \overline{P}^g = \overline{P}^R$.

**Remark 4.7.** By Proposition 4.8, the algebraic counterpart of the modal axiom K also holds in the above model of tqBa5 as $R^g = R$. Thus, it is also a model for the abstract algebra tqBa5 along with the modal axiom K.

**Rough Set model for a IA1**: For any equivalence relation $R^g$ on $U$ with $R^g_u = R^g_{g(u)}$, for all $u \in U$, by Proposition 4.1, Proposition 4.6, Propositions 4.7, Proposition 4.9, Proposition 4.10, Theorem 4.4 and Theorem 4.5, $\langle 2^U, \cap, \cup, \sim, I, C, \emptyset, U \rangle$ is a IA1, where $IP = \underline{P}_g = \underline{P}_R$ and $CP = \overline{P}^g = \overline{P}^R$.

**Remark 4.8.** It has been shown in Proposition 2.3 that the algebraic counterpart of the modal axiom K holds in a IA1. In the above model of IA1, the modal axiom K also holds (by Proposition 4.6 and Proposition 4.8).

A major objective of this paper is to obtain proper rough set models for the abstract algebras shown in Fig. 1 except for pre-rough algebra and qBa. In this paper we have successfully obtained such models for the algebras stqBa, system0, stqBa-D, stqBa-T, stqBa-B, tqBa, tqBa5 and IA1 only.

## 5. Concluding remarks

We may summarise the contents of this paper and indicate some future directions of work as follows.

- A number of abstract algebras whose core is qBa have been defined in order to build algebraic systems parallel to the modal systems T, TB, $TS_4$ and $TS_5$. These algebras are also suitably mapped with old algebras as shown in Fig. 1. As a result, a gap with respect to the standard modal systems in the previous studies of algebras may be considered as filled up.
- An algebraic form of the modal axiom K: $I(\neg x \vee y) \leqslant \neg Ix \vee Iy$ (quasi-Boolean base), relevant for all these algebras discussed in this sequel, is considered. It is found that this form of K is available in pre-rough algebra, IA1 and IA2 but not available in the remaining algebras stqBa, stqBa-D, stqBa-T, stqBa-B, tqBa, tqBa5, IA3, System0, SystemI and SystemII.
- Based on the availability of this form of K and the other standard modal axioms in the algebras, proper set theoretic rough set models of the algebras stqBa, stqBa-D, stqBa-T, stqBa-B, System0, tqBa, tqBa5 and IA1 have been constructed.
- The importance of these set models lies in their applicability in the domain of rough sets. It is true that no specific application has been shown but it is expected that in future some concrete cases will arise corresponding to some of the set models. Besides, it may be possible to establish the representation theorem of some of the algebras in terms of their set models and this is mathematically important. However, in this paper the idea has not been explored, it is a future possibility.
- For a fixed involution $g$, it is possible to obtain a collection of relations $\{R : R = R^g\}$ such that $\underline{P}_R$ and $\overline{P}^R$ are dual with respect to the complementation as well as the quasi-complementation. Resulting two algebras $\left\langle 2^U, \cap, \cup, ^c, \underline{P}_R, \overline{P}^R, \varnothing, U \right\rangle$ and $\left\langle 2^U, \cap, \cup, \sim, \underline{P}_R, \overline{P}^R, \varnothing, U \right\rangle$ are obtained, the first one is Boolean based whereas the last one is quasi-Boolean based. This observation may open up further study in the field of rough set theory. It is to be remembered that the notions of quasi-complementation and complementation do not coincide for an equivalence relation $R$ with $R = R^g$.

## CRediT authorship contribution statement

**Masiur Rahaman Sardar:** Conceptualization, Methodology, Writing - original draft, Writing - review & editing, Project administration. **Mihir Kumar Chakraborty:** Conceptualization, Methodology, Supervision.

## Data availability

No data was used for the research described in the article.

## Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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